RAN - 2203000205023001

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T.Y.B.Sc. (Sem. V) Examination October - 2023

Mathematics : MTH - 501

Group Theory

Time: 2 Hours] **[Total Marks: 50**

સૂચના : / Instructions

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નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the details of ☛ signs on your answer book

Name of the Examination:

☛ **T.Y.B.Sc. (Sem. V)**

Name of the Subject :

☛ **Mathematics : MTH - 501 Group Theory**

Subject Code No.: **2203000205023001**

Seat No.:

Student's Signature

(2) All questions are compulsory.
(3) Figures to the right indicate marks of corresponding question.
(4) Follow usual notations.
(5) Use of non-programmable scientific calculator is allowed.

1. Answer the following as directed: (Any FIVE) **(10)**

(1) If $x = x^{-1}$; for every element x in a group G, then G is abelian.

(2) Prove that a cyclic group is abelian.

(3) If for some elements a, b in a group G; $o(a)=3$ and $a \cdot b \cdot a^{-1} = b^2$, then find $o(b)$.

(4) Let H and K be subgroups of orders $o(H)$ and $o(K)$; respectively; of a finite group G ; where $o(H) > \sqrt{o(G)}$ and $o(K) > \sqrt{o(G)}$. Then prove that $H \cap K \neq (e)$.

(5) Define Euler's Function ϕ. State Euler's Theorem.

(6) Define Normal Subgroup. Give an example of a normal subgroup of an abelian group.

(7) If $\theta = (4\ 7\ 9\ 3)$ and $\sigma = (3\ 6\ 5)$ are the permutations in S_9, then find $\theta \cdot \sigma^{-1}$.

(8) Which of the following permutations in S_8 are even:
(a) $(3\ 5\ 8)\cdot(4\ 2\ 1\ 7)\cdot(8\ 7\ 6)$;(b) $(1\ 8\ 7\ 4\ 2\ 6)\cdot(8\ 2\ 5\ 3\ 4)\cdot(5\ 8)$.

2. Attempt any TWO: **(10)**

(a) Prove that:

(i) The identity element in a group G is unique.

(ii) If $(a\cdot b)^2 = a^2\cdot b^2$ for all a, b in a group G, then G is abelian.

(b) Prove that a non - empty subset H of a group G is a subgroup of G if and only if

(i) $a, b \in H \Rightarrow a\cdot b \in H$,

(ii) $a \in H \Rightarrow a^{-1} \in H$.

(c) Prove that the intersection of any two subgroups of a group is a subgroup.

3. Attempt any TWO: **(10)**

(a) Let H be any subgroup of a finite group G. Then prove that $o(H) \mid o(G)$.

(b) Prove that:

(i) If G is a finite group and a in G,then $a^{o(G)} = e$.

(ii) If p is a prime number and a is an integer such that $p \nmid a$, then $a^p \equiv a \pmod{p}$.

(c) If the order of a finite group G is a prime number p, then prove that G is cyclic.

4. Attempt any TWO: **(10)**

(a) Prove that a subgroup N of a group G is a normal subgroup of G if and only if $g \cdot N \cdot g^{-1} = N$; for every $g \in G$.

(b) Define Kernel Homomorphism of Groups. Let $\phi : G \to \overline{G}$ be a homomorphism of a group G in to a group $\overline{G}$. Then prove that:

(i) $\phi(e) = \overline{e}$, where e, $\overline{e}$ are the identity elements of groups G and $\overline{G}$ respectively;

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$; for every x in G.

(c) (i) Is the homomorphism $\phi : G \to G$; where $G = (a)$; a cyclic group of order 4; defined by $\phi(a) = a^2$; for every a in G; an automorphism of G ? Justify your answer,

(ii) Let $\phi : G \to G$ be an automorphism of a group G. Then prove that $o(\phi(a)) = o(a)$; for every a element in G.

5. **Attempt any TWO:** **(10)**

(a) (i) Find the Orbit of 7; for the permutation

$$\theta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 7 & 6 & 4 & 1 & 2 & 3 \end{bmatrix} \text{ in } S_7.$$

(ii) Verify whether $\tau^{-1} \cdot \omega$ is even or odd permutation; for the permutations $\tau = (1\ 7\ 2\ 9)$ and $\omega = (4\ 1\ 3)$ in S_9.

(b) Prove that there does not exist any permutation θ in S_9 satisfying $\theta^{-1} \cdot (3\ 7\ 9) \cdot \theta = (4\ 2\ 8\ 6)$.

(c) Given the permutations $\sigma = (1\ 2) \cdot (3\ 4)$ and $\tau = (3\ 1) \cdot (5\ 6)$ in S_6; find the permutation θ in S_6 satisfying $\theta^{-1} \cdot \sigma \cdot \theta = \tau$.

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